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Raman free-electron laser with transverse density gradients

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The influence of the electron density gradients on the free-electron laser in the Raman regime is considered. Since in the Raman regime the electron density appears in the resonance parameter, the density gradients prevent the beam from being wholly resonant. The transverse gradients can then substantially reduce the gain even if the total current is kept constant. We solve a model problem of a free-electron laser, which employs a sheet electron beam in a planar wiggler, and which operates in the small-signal, high-gain regime. We study two cases, a density profile of uniform density gradient and a general density profile. For a uniform density gradient we write a dispersion relation explicitly in terms of Whittaker functions. For general density profiles, providing the current or the wiggler field is small, we derive an approximate dispersion relation through a solvability condition of the perturbed equations. In both cases, when the density gradient is large, the gain is expressed as a sum of contributions from the individual resonant layers, where each such contribution is inversely proportional to the density gradient at the respective resonant layer. A sufficient condition in order for the gain not to be substantially reduced by the density gradients is formulated.

I. INTRODUCTION

The effects of the finite transverse dimensions on the free electron laser[1] (FEL) interaction have been studied extensively.[2–14] In a previous paper[13] we studied one of these effects, that of transverse density gradients on a small-signal, high-gain FEL which operates in the strong-pump regime. We solved a model problem of a sheet beam FEL in a planar wiggler and calculated the growth rate of the instability and the wave profile. We showed that when diffraction is large the solution is not sensitive to the density profile and depends on the total current only. When diffraction is small, density gradients change the growth rate of the instability but not in a substantial way. In the present article we explore the dependence of the interaction on the density gradients in a small-signal, high-gain FEL that operates in the Raman regime. In the Raman regime the influence of the density gradients is expected to be more dramatic since the density appears in the resonance denominator. A large transverse gradient causes this resonance denominator to vary transversely and prevents it from being small everywhere but at a particular layer, the resonant layer. Therefore, enlarging the transverse gradient would substantially reduce the gain in the FEL even if the total current were kept constant. We examine this issue by solving a model problem with a geometry similar to that of the model problem we studied earlier.[13]

In our model problem a rectangular, or sheet, electron beam propagates along a planar wiggler. Experiments with such a sheet-beam FEL in a planar wiggler have recently been reported.[15] The electron beam propagates in the z direction and is thin in the y direction, the direction of both the wiggler field and the transverse gradient of the wiggler field. The electron beam is much thicker in the x direction, which is possible practically because the planar wiggler has weak x dependence. We assume the sheet beam to be infinite in the x dimension and finite in the y dimension. We assume further that all the quantities are x independent and that all the transverse dependence is on y only. We write the linearized Maxwell and cold-fluid equations which are y and z dependent. We then simplify the equations near the FEL resonance in the Raman regime to be y dependent only. Finally, the governing equation is a second-order ordinary differential equation with an eigenvalue. The growth rate of the FEL instability is given by the imaginary part of the eigenvalue. For simplicity we assume that a conducting plate is located on each side of the sheet beam.

We find that in addition to the beam transverse density profile there are three characteristic parameters. Two of them appeared also in the strong-pump regime FEL:[3,10] a coupling parameter, which is a combination of the wiggler intensity and the electron current, and a detuning parameter. The third parameter measures the density gradient across the beam. In the Raman regime even if the density is constant to lowest order, this gradient parameter can be large and affect the interaction substantially. Our purpose here is to study the behavior of the solutions, the eigenvalues, and the eigenfunctions for large values of the gradient parameter.

In this paper we give results for an FEL in a slab geometry. We believe, however, that some of these results (such as the scaling of the gain for large density gradients) are not an artifact of the geometry, but apply more generally. This generalization of results is under current study.

In Sec. II we derive the governing equation and identify the three characteristic parameters. In Sec. III we study the case of a density profile with a uniform gradient. A dispersion relation is written in terms of Whittaker functions. Analytic expressions for the eigenvalue of the fundamental mode and for the wave field near the resonant layer are derived for large values of the gradient parameter, using the asymptotic forms of the Whittaker functions. The growth rate is shown to be inversely proportional to the gradient parameter. In Sec. IV we allow for a general density profile but limit ourselves to small values of the coupling parameter. In this case we expand the differential equation and write the dispersion relation in terms of a solvability condition. For large values of the gradient parameter we derive simple expressions for the eigenvalues and recover the results of Sec. III for the density profile with a uniform gradient. We emphasize that the change of the scaling of the growth rate may occur even for a beam of a density uniform to lowest order, since sometimes even a small relative density nonuniformity corresponds to a large gradient parameter. In Sec. V we examine the domain of validity of our asymptotic results. We also formulate a sufficient condition for the gain not to be reduced by density gradients. We then verify numerically our asymptotic results.

II. THE GOVERNING EQUATIONS

The derivation of the governing equation for the FEL interaction in the Raman regime is similar to the derivation for FEL interaction in the strong-pump regime.[13] We therefore present the derivation here in a condensed form.

A sheet electron beam, which is assumed to be infinitely wide in the x direction and of width $2a$ in the y direction, propagates in the z direction along a planar magnetic wiggler field of the approximated form

$$\mathbf{B}_0=\hat{\mathbf{e}}_y B_w \sin(k_0 z) \; . \tag{1}$$

We assume that $k_0 a$ is much smaller than 1 and neglect the transverse variations of the wiggler field. We look for the solutions of the cold-fluid equations for the electron normalized density H [the density multiplied by $4\pi e^2/(mc^2\gamma)$] and normalized momentum $\mathbf{P}$ (the momentum divided by mc). Here e and m are the electron charge and mass, c is the velocity of light in vacuum, and γ is $(1+\mathbf{P}\cdot\mathbf{P})^{1/2}$. The normalized electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ [the fields multiplied by $e/(mc^2)$] are described by solutions of the Maxwell equations. The cold-fluid equations are the continuity equation

$$\frac{1}{c}\frac{\partial}{\partial t}(\gamma H)+\nabla\cdot(\mathbf{P}H)=0 \; , \tag{2}$$

and the momentum equation

$$\frac{\gamma}{c}\frac{\partial \mathbf{P}}{\partial t}+\mathbf{P}\cdot\nabla\mathbf{P}=-\gamma\mathbf{E}-\mathbf{P}\times\mathbf{B} \; . \tag{3}$$

The Maxwell equations are

$$\nabla\times\mathbf{B}=\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}-\mathbf{P}H \; , \tag{4}$$

$$\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \; , \tag{5}$$

$$\nabla\cdot\mathbf{E}=-\gamma H \; . \tag{6}$$

In order to perform a linear stability analysis we write each quantity as

$$F(y,z,t)=f(y,z)+\mathrm{Re}\left[e^{i(kz-\omega ct)}\sum_{n=-\infty}^{\infty}\delta f_n(y)e^{ink_0 z}\right] , \tag{7}$$

where $f(y,z)$ is a zeroth-order time-independent term and $\delta f_n(y)$ is the amplitude of the Fourier component of the first-order time-dependent term. The time-independent solutions of the cold-fluid equations for the time-independent energy $\Gamma(y,z)$, momentum $\mathbf{p}(y,z)$, and density $h(y,z)$ are

$$\Gamma(y,z)=\Gamma=\text{const} \; , \tag{8a}$$

$$\mathbf{p}=-\hat{\mathbf{e}}_x\frac{B_w}{k_0}\cos(k_0 z)+\hat{\mathbf{e}}_z(\Gamma^2-1)^{1/2} \; , \tag{8b}$$

$$h(y,z)=h(y) \; . \tag{8c}$$

We assumed that B_w^2/k_0^2 is much smaller than (Γ^2-1), and being interested in the fundamental resonance only, we neglected the small z modulation of p_z and h. We also neglected static self-fields of the beam.

We look for k close to ω and assume that the largest Fourier components of the wave fields are $\delta\mathbf{E}_0$ and $\delta\mathbf{B}_0$. We assume that the density h is high enough so that we cannot neglect δE_z. Linearizing the continuity equation [Eq. (2)], the z component of the momentum equation [Eq. (3)] and Poisson's equation [Eq. (6)], we obtain the following coupled equations for $\delta E_{z,1}$, $\delta p_{z,1}$, and δh_1:

$$p_z\Delta\delta h_1=h(\omega p_z/\Gamma-k-k_0)\delta p_{z,1} \; , \tag{9a}$$

$$ip_z\Delta\delta p_{z,1}=\frac{B_w}{2}\left[\frac{\delta B_{y,0}}{k_0}+i\delta p_{x,0}\right]-\Gamma\delta E_{z,1} \; , \tag{9b}$$

$$i(k+k_0)\delta E_{z,1}=-\frac{hp_z}{\Gamma}\delta p_{z,1}-\Gamma\delta h_1 \; , \tag{9c}$$

where

$$\Delta\equiv-\omega\Gamma/p_z+(k+k_0) \; . \tag{9d}$$

Note that in Poisson's equation we neglected the derivative of $\delta E_{y,1}$ with respect to y since we assumed that the derivative of $\delta E_{z,1}$ with respect to z is larger. We also required that $(\sqrt{2}\Gamma k_0 a)$ be much larger than 1. If this requirement is not satisfied, the form of the equations is modified due to a transverse dependence of space-charge effects.[12,14] Solving this set of equations [9(a)–9(c)], we obtain for the perturbed density

$$\delta h_1=\frac{2iS^2B_w\delta E_{x,0}}{(\Delta^2-S^2)} \; , \tag{10}$$

where

$$S^2\equiv(h/p_z^2)(1+B_w^2/2k_0^2) \; .$$

We require $|\Delta - S|$ to be much smaller than S and

$$\mathrm{Im}(k) \ll S \ . \tag{11}$$

These are conditions for operating in the Raman regime. To lowest order the perturbed current is

$$\delta j = -\hat{e}_x p_x \delta h_1 \ , \tag{12}$$

and thus Maxwell's equations (4) and (5) become

$$\frac{\partial^2 \delta E_{x,0}}{\partial \bar{y}^2} + (\omega^2 - k^2) a^2 \delta E_{x,0} = \frac{B_w^2 \omega S a^2 \delta E_{x,0}}{2k_0(\Delta - S)} \ . \tag{13}$$

We defined a new independent variable $\bar{y} \equiv y/a$ and used inequality (11).

Equation (13) governs the FEL interaction in the Raman regime with a nonuniform beam density. The only equilibrium y-dependent quantity in Eq. (13) is the density h and consequently S. With appropriate boundary conditions this governing equation could be solved for the eigenvalue k (which appears also in Δ) and for the eigenfunction $\delta E_{x,0}$. We will first reduce the number of parameters through a change of variables and through the introduction of nondimensional parameters. In its final form the governing equation will have three characteristic parameters only.

We are mainly interested in the influence of the density gradients on the interaction, and write the density as

$$h(\bar{y}) = h_0 + \hat{\mu} f(\bar{y}) \ . \tag{14}$$

We require that $f(\bar{y})$ satisfies

$$\int_{-1}^{1} d\bar{y}\, f(\bar{y}) = 0 \ , \tag{15a}$$

and that the difference between the maximal and the minimal values of $f(\bar{y})$ is 1,

$$f_{\max} - f_{\min} = 1 \ . \tag{15b}$$

The average density is h_0. We also require that

$$\hat{\mu} \ll h_0 \ . \tag{16}$$

Thus the density is constant to lowest order. We define the eigenvalue ν

$$\nu \equiv k - \omega \ , \tag{17}$$

and thus

$$\Delta - S = \nu + \hat{\xi} - \bar{\mu} f(\bar{y}) \ , \tag{18}$$

where

$$\bar{\mu} \equiv (\hat{\mu}/2)(h_0 p_z^2)^{-1/2} \left[1 + \frac{B_w^2}{2k_0^2} \right]^{1/2} , \tag{19}$$

and $\hat{\xi}$, the mismatch parameter, is

$$\hat{\xi} = \omega(1 - \Gamma/p_z) + k_0 - (h_0/p_z^2)^{1/2} \left[1 + \frac{B_w^2}{2k_0^2} \right]^{1/2} . \tag{20}$$

We assume that $\hat{\xi}$ is zero to lowest order, in which case the frequency is the resonant frequency

$$\omega \cong \frac{1}{(\Gamma/p_z - 1)} \left[k_0 - (h_0/p_z^2)^{1/2} \left[1 + \frac{B_w^2}{2k_0^2} \right]^{1/2} \right] . \tag{21}$$

For a relativistic beam and when S is much smaller than k_0, the resonant frequency becomes

$$\omega \cong 2k_0 \Gamma^2 \ . \tag{22}$$

We approximate $(\omega + k)$ as 2ω, define a nondimensional eigenvalue

$$\phi^2 \equiv -2\omega\nu a^2 \ , \tag{23}$$

and write Eq. (13) in a form equivalent to the form of the equation in the strong-pump regime[5,13] with similar notations,

$$\frac{\partial^2 \delta E_x}{\partial \bar{y}^2} + \phi^2 \delta E_x + \frac{\alpha \delta E_x}{[\phi^2 - \bar{\xi} + \mu f(\bar{y})]} = 0 \ . \tag{24}$$

The governing equation now contains three characteristic parameters only; a coupling parameter

$$\alpha \equiv \frac{B_w^2 \omega^2}{k_0} \left[\frac{h}{p_z^2} \left[1 + \frac{B_w^2}{2k_0^2} \right] \right]^{1/2} a^4 \ , \tag{25a}$$

a detuning parameter

$$\bar{\xi} \equiv 2\omega a^2 \hat{\xi} \ , \tag{25b}$$

and a gradient parameter

$$\mu \equiv 2\omega a^2 \bar{\mu} \tag{25c}$$

which measures the rate of the density nonuniformity. In the expression for α we approximated the density with h_0, and thus α is constant. For notational convenience we suppressed the subscript 0 in $\delta E_{x,0}$. When the relation (13b) is used, α may be approximated as

$$\alpha \cong 4k_0 B_w^2 \Gamma^3 [h(1 + B_w^2/2k_0^2)]^{1/2} a^4 \ . \tag{26}$$

All the $\bar{y}$ dependence of the equilibrium enters in the denominator of the third term of Eq. (24). Note that we could neglect the density gradients in the numerator because of inequality (16) but not in the resonant denominator which may take small values. Note also from the expressions (19) and (25b), that the gradient parameter μ can be large even if the relative density nonuniformity $(\hat{\mu}/h_0)$ is small.

We turn now to the boundary conditions. We will study density profiles symmetric across the midplane so that

$$f(\bar{y}) = f(-\bar{y}) \ . \tag{27}$$

Assuming also symmetric boundary conditions we will look for symmetric solutions which satisfy

$$\frac{\partial \delta E_x}{\partial \bar{y}} = 0 \text{ at } \bar{y} = 0 \tag{28}$$

or antisymmetric solutions which obey

$$\delta E_x = 0 \text{ at } \bar{y} = 0 \ . \tag{29}$$

For simplicity we limit ourselves to solutions which satisfy

$$\delta E_x = 0 \text{ at } \bar{y} = 1 , \tag{30}$$

which follows a perfectly conducting waveguide located at $\bar{y}=1$.

III. A DENSITY PROFILE WITH UNIFORM GRADIENT

We look for nonreal eigenvalues ϕ^2 with a positive imaginary part corresponding to growing waves. We focus on the dependence of the eigenfunction δE_x and the eigenvalue ϕ^2 on various values of the gradient parameter μ.

We start with the case of uniform density. When

$$\mu = 0 , \tag{31}$$

the symmetric solution is

$$\delta E_x = \cos(\chi \bar{y}) , \tag{32a}$$

and the antisymmetric solution is

$$\delta E_x = \sin(\chi \bar{y}) , \tag{32b}$$

where

$$\chi^2 \equiv \phi^2 + \frac{\alpha}{(\phi^2 - \bar{\xi})} . \tag{33}$$

Following (30) the sets of allowed χ^2 satisfy

$$(\chi^2)_n = \left[\frac{\pi}{2} \right]^2 (2n+1)^2, \quad n = 0,1,2,\ldots \tag{34a}$$

for the symmetric solution, and

$$(\chi^2)_n = (\pi n)^2, \quad n = 1,2,3,\ldots \tag{34b}$$

for the antisymmetric solution. Two eigenvalues ϕ^2 correspond to each value of χ^2, following Eq. (33). If the mismatch parameter is such that

$$[\bar{\xi} - (\chi^2)_n]^2 < 4\alpha , \tag{35}$$

the two eigenvalues become nonreal and complex conjugates. In particular, when

$$\bar{\xi} = (\chi^2)_n \tag{36}$$

the imaginary part of the eigenvalue is, as in the 1D case,

$$\mathrm{Im}\phi^2 = \alpha^{1/2} . \tag{37}$$

We turn now to the case of nonzero μ. In most of this paper we examine a particularly simple density profile which enables us to derive analytic results. Let us assume that

$$f(\bar{y}) = \tfrac{1}{2} - \bar{y}, \quad 1 \ge \bar{y} \ge 0 . \tag{38}$$

The corresponding density profile is described by curve 1 in Fig. 1. By varying μ we do not vary the total current of the beam and the average density remains h_0. The governing equation (24) becomes now

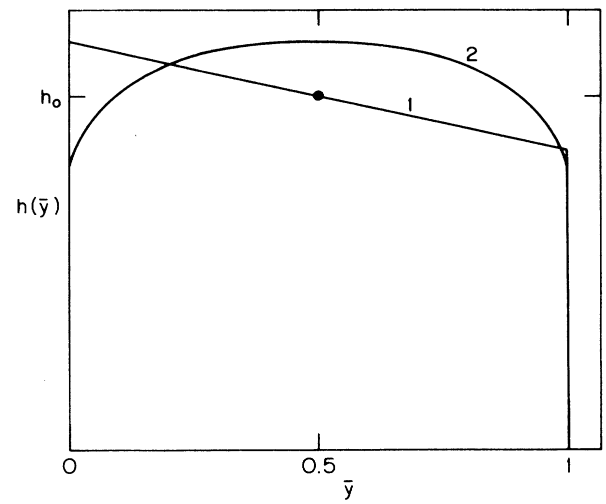

FIG. 1. Two density profiles: (1) constant gradient, (2) parabolic profile.

$$\frac{\partial^2 \delta E_x}{\partial \bar{y}^2} + \phi^2 \delta E_x + \frac{\alpha}{[\phi^2 - \bar{\xi} - \mu(\bar{y} - \frac{1}{2})]} \delta E_x = 0 . \tag{39}$$

This equation is transformed into the Whittaker equation

$$\frac{\partial^2 \delta E}{\partial \xi^2} + \left[-\frac{1}{4} + \frac{\kappa}{\xi} \right] \delta E = 0 , \tag{40}$$

where

$$\kappa \equiv \frac{i\alpha}{2\phi\mu} , \tag{41a}$$

$$\xi \equiv 2i\phi(\bar{y} - \tfrac{1}{2} + \zeta) - 2i\phi^3 \mu , \tag{41b}$$

$$\zeta \equiv \bar{\xi}/\mu , \tag{41c}$$

and

$$\delta E(\xi) \equiv \delta E_x(\bar{y}) . \tag{41d}$$

The general solution of Eq. (40) is

$$\delta E \equiv A W_{\kappa,1/2}(\xi) + B W_{-\kappa,1/2}(-\xi) . \tag{42}$$

The boundary conditions are now

$$A \frac{dW_{\kappa,1/2}(\xi)}{d\xi} \Bigg|_{\xi=\xi_0} + B \frac{dW_{-\kappa,1/2}(-\xi)}{d\xi} \Bigg|_{\xi=\xi_0} = 0 , \tag{43a}$$

or

$$A W_{\kappa,1/2}(\xi_0) + B W_{-\kappa,1/2}(-\xi_0) = 0 , \tag{43b}$$

and

$$A W_{\kappa,1/2}(\xi_1) + B W_{-\kappa,1/2}(-\xi_1) = 0 . \tag{44}$$

Here ξ_0 and ξ_1 are

$$\xi_0 \equiv 2i\phi(-\tfrac{1}{2} + \zeta) - 2i\phi^3/\mu , \tag{45a}$$

$$\xi_1 \equiv 2i\phi(\tfrac{1}{2}+\zeta)-2i\phi^3/\mu \ . \tag{45b}$$

The dispersion relation for the symmetric solution, following (43a) and (44), is

$$W_{-\kappa,1/2}(-\xi_1)\frac{dW_{\kappa,1/2}(\xi)}{d\xi}\Bigg|_{\xi=\xi_0} - W_{\kappa,1/2}(\xi_1)\frac{dW_{-\kappa,1/2}(-\xi)}{d\xi}\Bigg|_{\xi=\xi_0}=0 \ . \tag{46}$$

For the antisymmetric solution, following (43b) and (44), the dispersion relation is

$$W_{-\kappa,1/2}(-\xi_1)W_{\kappa,1/2}(\xi_0) - W_{\kappa,1/2}(\xi_1)W_{-\kappa,1/2}(-\xi_0)=0 \ . \tag{47}$$

We turn now to study the effects of density gradients on the interaction. Since $\arg(\phi^2)$ is smaller than π, ϕ is located in the first quadrant in the complex plane. When μ is large so that

$$\frac{\phi^2}{\mu} \ll 1 \ , \tag{48}$$

and if $|\zeta|<\frac{1}{2}$, the line of integration starts in the fourth quadrant of the complex ξ plane, passes through the first quadrant, and ends in the second quadrant as shown in Fig. 2. The line crosses the real axis at

$$\xi_r = \frac{4|\phi|^2\phi_i}{\mu} \ , \tag{49a}$$

and the imaginary axis at

$$\xi_i = \frac{4|\phi|^2\phi_r}{\mu} \ . \tag{49b}$$

The line that connects ξ_0 and ξ_1 in the ξ plane is shown in Fig. 2. We chose the Riemann surfaces such that

$$-\frac{\pi}{2} < \arg(\xi_0) < 0, \quad \frac{\pi}{2} < \arg(\xi_1) < \pi \ , \tag{50}$$

and we pick

$$\arg(-\xi) = \arg(\xi) - \pi \ . \tag{51}$$

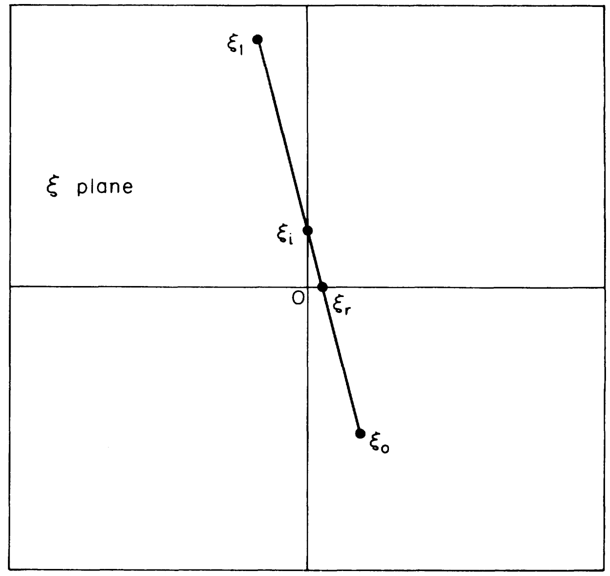

FIG. 2. Path of integration in the complex ξ plane for large μ.

Thus $|\arg(-\xi_0)|$ is greater than π, and we use the identity[16]

$$W_{-\kappa,1/2}(-\xi_0) = e^{2\pi i\kappa}W_{-k,1/2}(\xi_0 e^{i\pi}) - \frac{2\pi i e^{\pi i\kappa}W_{\kappa,1/2}(\xi_0)}{\Gamma(\kappa)\Gamma(1+\kappa)} \ . \tag{52}$$

When μ is large, κ is small. For ξ not near the origin we may approximate

$$W_{\kappa,1/2}(\xi) = e^{-\xi/2}[1+\kappa\ln(\xi)+\kappa F(\xi)] \ , \tag{53}$$

where

$$F(\xi) \equiv \int_{\xi}^{\infty} e^{(\xi-\xi')}\frac{d\xi'}{\xi'} \ . \tag{54}$$

The dispersion relation (46) becomes, when use is made of Eqs. (52)–(53),

$$e^{(\xi_1-\xi_0)}\{-1+\kappa[\ln(\xi_1/\xi_0)-i\pi+F(\xi_0)+F(-\xi_1)]\} - \{1+\kappa[\ln(\xi_1/\xi_0)+i\pi+F(-\xi_0)+F(\xi_1)]\} - 2\pi i\kappa e^{-\xi_0} = 0 \ . \tag{55}$$

We limit ourselves to the fundamental mode. To zeroth order in $1/\mu$ the dispersion relation yields

$$\phi^{(0)} = \pi/2 \ , \tag{56a}$$

$$\xi_0^{(0)} = \pi(\tfrac{1}{2}-\zeta)e^{-i\pi/2} \ , \tag{56b}$$

and

$$\xi_1^{(0)} = \pi(\tfrac{1}{2}+\zeta)e^{i\pi/2} \ . \tag{56c}$$

Expanding the dispersion relation to first order in $1/\mu$, we obtain

$$\phi^{(1)} = 2\frac{i\alpha}{\mu}\cos^2\left[\frac{\pi}{2}(\zeta-\tfrac{1}{2})\right] + \frac{\alpha}{\mu}\left[\cos(\pi\zeta)+\frac{G(\zeta)}{\pi}\right] \ , \tag{57}$$

where

$$G(\zeta) \equiv \sin(\pi\zeta)\mathrm{Re}[E_1(i\pi(\tfrac{1}{2}-\zeta))-E_1(i\pi(\tfrac{1}{2}+\zeta))] + \cos(\pi\zeta)[\pi - S_i(\pi(\tfrac{1}{2}-\zeta)) - S_i(\pi(\tfrac{1}{2}+\zeta))] - \ln(\eta) \ . \tag{58a}$$

Here

$$E_1(z)=\int_z^{\infty} dt\, e^{-t}/t \tag{58b}$$

is the exponential integral, and

$$S_i(\alpha)=\int_0^{\alpha} dt\, \sin t/t \tag{58c}$$

is the sine integral. Also

$$\eta \equiv \left[\frac{1+2\zeta}{1-2\zeta}\right] . \tag{58d}$$

Finally, the imaginary part of the normalized eigenvalue is

$$\mathrm{Im}\phi^2=\left[\frac{2\pi\alpha}{\mu}\right]\cos^2\left[\frac{\pi}{2}(\zeta-\tfrac{1}{2})\right] \tag{59}$$

and is proportional to α and not to $\alpha^{1/2}$ as it is in the case of uniform density. Note also that

$$y_R \equiv \zeta-\tfrac{1}{2} \tag{60}$$

is the approximate location of the resonant layer, so that when $\bar{y}$ equals y_R, ξ is zero to lowest order in $1/\mu$. When y_R is zero, the eigenvalue has its maximum imaginary part. On the other hand, when y_R is one, the resonant layer is at the beam boundary where the wave vanishes, as does the imaginary part of the eigenvalue.

We turn now to the behavior of the solution near the resonant layer. From Eq. (43b) we find that to lowest order

$$B/A=e^{-i\pi(\zeta-1/2)} . \tag{61}$$

From Eqs. (41a), (49b), (56a), and (57) we note that near the resonant layer $|\xi|$ is much smaller than $|\kappa|$. Using the approximate form of the Whittaker function with small argument[17]

$$W_{\kappa,1/2}(\xi)=e^{-\xi/2}\left[\frac{\xi\ln\xi}{\Gamma(-\kappa)}+\frac{1}{\Gamma(1-\kappa)}\right] \tag{62}$$

and the smallness of κ, we obtain the field near the origin

$$\delta E=(1+e^{-i\pi(\zeta-1/2)})(1-\kappa\xi\ln\xi) . \tag{63}$$

The electric field is smooth near the resonant layer. The magnetic component of the wave δB_z, which is proportional to the derivative of δE, has logarithmic singularity near the resonant layer. However, at a given $\bar{y}$, when μ is increased, the magnetic field remains bounded since $\kappa\ln\xi$ goes to zero.

For completeness we repeat the analysis for the antisymmetric solution. The dispersion relation for the antisymmetric soltuion (47), becomes, when Eqs. (52) and (53) are used,

$$e^{(\xi_1-\xi_0)}\{1+\kappa[-\ln(\xi_1/\xi_0)+i\pi+F(\xi_0)-F(-\xi_1)]\}-\{1+\kappa[\ln(\xi_1/\xi_0)+i\pi-F(-\xi_0)+F(\xi_1)]\}-2\pi i\kappa e^{-\xi_0}=0 . \tag{64}$$

To zeroth order, the eigenvalue of the fundamental antisymmetric mode is

$$\phi^{(0)}=\pi \tag{65}$$

and

$$\xi_0^{(0)}=2\pi(\tfrac{1}{2}-\zeta)e^{-i\pi/2} , \tag{66a}$$

$$\xi_1^{(0)}=2\pi(\tfrac{1}{2}+\zeta)e^{i\pi/2} . \tag{66b}$$

Expanding the dispersion relation (64) to first order in $1/\mu$, we now obtain

$$\phi^{(1)}=\frac{i\alpha}{\mu}\sin^2[\pi(\zeta-\tfrac{1}{2})]+\frac{\alpha}{2\mu}\left[-\sin(2\pi\zeta)+\frac{R(\zeta)}{2\pi}\right] , \tag{67}$$

where

$$R(\zeta)=F(\xi_0)+F(-\xi_0)-F(\xi_1)-F(-\xi_1)-2\ln\eta . \tag{68}$$

The imaginary part of the normalized eigenvalue is

$$\mathrm{Im}\phi^2=\left[\frac{2\pi\alpha}{\mu}\right]\sin^2[\pi(\zeta-\tfrac{1}{2})] . \tag{69}$$

The growth rate of the instability is the same as that of the symmetric solution, except for the factor $\sin^2(\pi y_R)$ instead of $\cos^2(\pi/2y_R)$ [Eq. (59)]. The solution near the resonant layer is found here to be

$$\delta E=[1-e^{-2\pi(\zeta-1/2)}](1-\kappa\xi\ln\xi) . \tag{70}$$

In summary, in this section we derived expressions for the eigenfunctions and dispersion relations for the eigenvalues in the case of a beam with a uniform density gradient. For a large gradient parameter, we gave an analytic expression for the imaginary part of the eigenvalue, which is different from the expression given in the case of a uniform density beam. The growth rate of the instability in the nonuniform density case is proportional to the ratio of the coupling parameter to the gradient parameter, in contrast to the case of a uniform density where it is proportional to the square root of the coupling parameter. We emphasize that the change of the scaling of the growth rate may occur even for a beam of a density uniform to lowest order, since sometimes even a small density nonuniformity corresponds to a large gradient parameter.

IV. GENERAL DENSITY PROFILE

It is possible to obtain some analytic results for an electron beam of a general density profile when the density gradient is large, if α is small. We obtain these results by solving the governing equation by a perturbation method. We derive a solvability condition for the perturbed equation which comprises a dispersion relation. We used a similar method previously in the study of thick beam FEL.[14]

We go back to Eq. (24), and assuming that the third term is small, we write it to lowest order as

$$\frac{\partial^2 \delta E_x^{(0)}}{\partial \bar{y}^2} + \phi^{2^{(0)}} \delta E_x^{(0)} = 0 \ . \tag{71}$$

The fundamental symmetric and antisymmetric solutions are

$$\delta E_x^{(0)} = \cos(\phi^{(0)} \bar{y}) \ , \quad \phi^{(0)} = \pi/2 \ , \tag{72a}$$

and

$$\delta E_x^{(0)} = \sin(\phi^{(0)} \bar{y}) \ , \quad \phi^{(0)} = \pi \ , \tag{72b}$$

respectively. We now write Eq. (24) to first order

$$\frac{\partial^2 \delta E_x^{(1)}}{\partial \bar{y}^2} + \phi^{2^{(0)}} \delta E_x^{(1)} = Q \equiv -\phi^{2^{(1)}} \delta E_x^{(0)} - \frac{\alpha \delta E_x^{(0)}}{[\phi^{2^{(0)}} + \phi^{2^{(1)}} - \bar{\xi} + \mu f(\bar{y})]} \ . \tag{73}$$

This is an inhomogeneous equation for $\delta E_x^{(1)}$ and its general solution is

$$\delta E_x^{(1)} = -\frac{1}{\phi^{(0)}} \left[\int_0^{\bar{y}} d\bar{y}' \sin(\phi^{(0)} \bar{y}') \cos(\phi^{(0)} \bar{y}) Q(\bar{y}') + \int_{\bar{y}}^1 d\bar{y}' \sin(\phi^{(0)} \bar{y}) \cos(\phi^{(0)} \bar{y}') Q(\bar{y}') \right] + \bar{A} \cos(\phi^{(0)} \bar{y}) + \bar{B} \sin(\phi^{(0)} \bar{y}) \ . \tag{74}$$

First we look at the symmetric solution. Since we want the solution to vanish at $\bar{y}=1$, we require $\bar{B}=0$. For the symmetric solution the derivative has to vanish at $\bar{y}=0$ and thus

$$\int_0^1 d\bar{y}' \cos(\phi^{(0)} \bar{y}') Q(\bar{y}') = 0 \ . \tag{75}$$

The antisymmetric solution vanishes at the origin and thus $\bar{A}=0$. The requirement that the antisymmetric solution also vanish at $\bar{y}=1$ yields

$$\int_0^1 d\bar{y}' \sin(\phi^{(0)} \bar{y}') Q(\bar{y}') = 0 \ . \tag{76}$$

The solvability conditions (75) and (76) comprise the dispersion relations. These are

$$\int_0^1 d\bar{y}' \cos^2\left[\frac{\pi}{2}\bar{y}'\right] \left[\phi^{2^{(1)}} + \frac{\alpha}{[(\pi/2)^2 + \phi^{2^{(1)}} - \bar{\xi} + \mu f(\bar{y}')]}\right] = 0 \tag{77}$$

for the symmetric solution, and

$$\int_0^1 d\bar{y}' \sin^2(\pi \bar{y}') \left[\phi^{2^{(1)}} + \frac{\alpha}{[\pi^2 + \phi^{2^{(1)}} - \bar{\xi} + \mu f(\bar{y}')]}\right] = 0 \tag{78}$$

for the antisymmetric solution.

Further analytic progress is possible when the density gradient is large. When μ is large, $(\mathrm{Im}\phi^{2^{(1)}})/\mu$ becomes small, and Eq. (77) becomes

$$\tfrac{1}{2}\phi^{2^{(1)}} - \frac{i\pi\alpha}{\mu} \sum_j \frac{\cos^2\left[\frac{\pi}{2}\bar{y}_{R,j}\right]}{|f'(\bar{y}_{R,j})|} + \mathrm{P}\int_0^1 d\bar{y}' \cos^2\left[\frac{\pi}{2}\bar{y}'\right] \frac{\alpha}{[(\pi/2)^2 + \mathrm{Re}(\phi^{2^{(1)}}) - \bar{\xi} + \mu f(\bar{y}')]} = 0 \ . \tag{79}$$

Here $\bar{y}_{R,j}$ are the roots of

$$(\pi/2)^2 + \mathrm{Re}(\phi^{2^{(1)}}) - \bar{\xi} + \mu f(\bar{y}_{R,j}) = 0 \ , \tag{80}$$

which satisfy $0 < \bar{y}_{R,j} < 1$. If $\mathrm{Re}(\phi^{2^{(1)}})$ is smaller than $[(\pi/2)^2 - \bar{\xi}]$, and when $f(\bar{y})$ takes the form in Eq. (38), Eq. (59) is recovered. The real part of the eigenvalue then becomes

$$\mathrm{Re}(\phi^{2^{(1)}}) = -\int_0^1 d\bar{y}' [1 + \cos(\pi \bar{y}')] \frac{\alpha}{[(\pi/2)^2 - \bar{\xi} - \mu(\bar{y}' - \frac{1}{2})]} \ , \tag{81}$$

which can be shown to be identical to the form of the real part in Eqs. (57) and (58a). Similarly, for the antisymmetric solution we obtain

$$\tfrac{1}{2}(\phi^2)^{(1)} - \frac{i\pi\alpha}{\mu} \sum_j \frac{\sin^2(\pi \bar{y}_{R,j})}{|f'(\bar{y}_{R,j})|} + \mathrm{P}\int_0^1 d\bar{y}' \sin^2(\pi \bar{y}') \frac{\alpha}{[\pi^2 + \mathrm{Re}(\phi^{2^{(1)}}) - \bar{\xi} + \mu f(\bar{y}')]} = 0 \ , \tag{82}$$

where $\bar{y}_{R,j}$ are here the roots of

$$\pi^2+\mathrm{Re}(\phi^{2^{(1)}})-\bar{\xi}+\mu f(\bar{y}_{R,j})=0 , \quad (83)$$

which satisfy $0<\bar{y}_{R,j}<1$. Again these expressions could be shown to agree with (67) and (68). One should note that Eqs. (79) and (82) are correct only if $f'(\bar{y}_{R,j})$ is not too small.

In summary, in this section we showed that when the coupling parameter is small, the eigenvalue can be found from the solution of an integral equation [Eq. (65) or (66)] in which the eigenvalue is the only unknown. Moreover, if, in addition, the gradient parameter is large, the imaginary part of the eigenvalue is expressed analytically as a sum of contributions from the resonant layers [Eq. (79) or (82)].

V. DISCUSSION AND NUMERICAL EXAMPLES

Let us start by discussing the domain of validity of our results. Throughout the derivation we required the following inequalities to be satisfied:

$$\mathrm{Im}(k)\ll S$$

and

$$\hat{\mu}\ll h_0 ,$$

which are Eqs. (11) and (16), respectively. We now require that inequality (11) be satisfied even for the uniform density beam, when the imaginary part of the eigenvalue is maximal and equals $\alpha^{1/2}$. Using the definition of S and the expression (26) for α, we express inequality (11) in this case as

$$\frac{\alpha^{1/2}}{2}\ll G , \quad (84)$$

where

$$G\equiv 2\Gamma(k_0 a)^2\left[\frac{h_0}{k_0^2}\left[1+\frac{B_w^2}{2k_0^2}\right]\right]^{1/2} . \quad (85)$$

Inequality (84), if satisfied, assures us that the FEL operates in the Raman regime. We now use the definitions (19) and (25b), as well as Eq. (22), and write inequality (16) as

$$\mu\ll G . \quad (86)$$

When inequalities (11) and (16), or equivalently (84) and (86), are satisfied, Eq. (24) governs the interaction. Thus, for any pair of dimensionless parameters α and μ, we may choose physical parameters such that the quantity G satisfies (84) and (86), and consequently the governing equation (24) holds. Note also that we assume that the FEL resonance condition is satisfied.

In order that the gain be described by the asymptotic results of Sec. III, it must be reduced substantially by the density gradient. The gain, when the gradient parameter is large, $2\pi\alpha/\mu$, must be much smaller than the gain when the density is uniform, $\alpha^{1/2}$. We thus require

$$2\pi\alpha^{1/2}\ll\mu . \quad (87)$$

If both (86) and (87) are satisfied, inequality (84) is satisfied automatically. For the validity of our asymptotic results, it is sufficient that

$$2\pi\alpha^{1/2}\ll\mu\ll G . \quad (88)$$

For operation in the Raman regime, G must be much larger than $\alpha^{1/2}/2$, (84). However, for the validity of our model, there is a stronger requirement, that G be much larger than $2\pi\alpha^{1/2}$. If G is much larger than $2\pi\alpha^{1/2}$, inequality (88) determines the values of the gradient parameter for which the asymptotic results hold. Note that

$$\mu=G(\hat{\mu}/h_0) , \quad (89)$$

and therefore G is the maximal possible value of μ. Thus, if G is not much larger than $2\pi\alpha^{1/2}$, the density gradient of the beam could not cause the drastic reduction of the gain which was described by the asymptotic results of Sec. III. We conclude from (84) and (88) that if

$$1\ll 2G/\alpha^{1/2}\leq 4\pi , \quad (90)$$

the FEL operates in the Raman regime and the gain reduction due to density gradients cannot be drastic. We regard inequality (90) as a sufficient condition in order for the gain not to be substantially reduced by density gradients. If, however,

$$4\pi\ll 2G/\alpha^{1/2} , \quad (91)$$

any density gradient which satisfies

$$4\pi\ll 2\mu/\alpha^{1/2} \quad (92)$$

as well will cause gain reduction as described by our asymptotic results.

It is useful to write these conditions in terms of the physical parameters of the problem. Our asymptotic results hold if the following inequalities are satisfied:

$$4\pi U\ll\hat{\mu}/h_0\ll 1 , \quad (93)$$

where

$$U\equiv\tfrac{1}{2}(B_w/k_0)\left[\frac{h_0}{k_0^2\Gamma^2}\left[1+\frac{B_w^2}{2k_0^2}\right]\right]^{-1/4} . \quad (94)$$

Equation (93) is obtained by substituting into (88) the definitions of α, μ, and G. The sufficient condition for operation in the Raman regime with no drastic gain reduction due to the density gradients [Eq. (90)] becomes, in terms of the physical parameters,

$$1/(4\pi)\leq U\ll 1 . \quad (95)$$

If, however,

$$U\ll 1/(4\pi) , \quad (96)$$

the density gradient which satisfies inequality (93) will cause drastic reduction in the gain.

For completeness we write here also the relations

$$\alpha=4G^2U^2 \quad (97)$$

and

$$\mu=G(\hat{\mu}/h_0) . \quad (98)$$

For a given set of physical parameters one may calculate U and find whether one of the inequalities (95) or (96) is satisfied. One could then calculate G and $\hat{\mu}/h_0$ and find the characteristic parameters α and μ using (97) and (98).

We turn now to numerical examples. Figure 3 shows $\mathrm{Im}\phi^2$ as a function of the gradient parameters μ for $\bar{\zeta}(\equiv\zeta\mu)$ equals $(\pi/2)^2$ (curve 1) and for ζ equals 0.5 (curve 2). In this figure α is 0.5. The dashed line shows the asymptotic behavior [Eq. (59)]. There is good agreement between the analytic and computed values. When μ is small the maximum of $\mathrm{Im}\phi^2$ occurs for $\bar{\zeta}$ close to $(\pi/2)^2$, while for large μ, the maximum of $\mathrm{Im}\phi^2$ is for ζ close to 0.5. In both curves the results are for the fundamental symmetric solution.

In Fig. 4 we examine the growth rate of the instability for a density profile whose function $f(\bar{y})$ [see Eq. (14)] is

$$f(\bar{y})=\tfrac{1}{3}-4(\bar{y}-\tfrac{1}{2})^2 . \quad (99)$$

This function is described by curve 2 in Fig. 1. The gradient parameter μ is 100 and α is 0.5 as before. Curve 2 in Fig. 4 shows the growth rate $\mathrm{Im}(\phi^2)$ as a function of the detuning parameter and as calculated by solving the full differential equations (24). The solution is the fundamental symmetric solution. Curve 1 shows the value $\bar{y}_{R,1}$ of the resonant plane as a function of the detuning parameter and as found from Eq. (83). The value of the real part of the eigenvalue that was used for calculating $\bar{y}_R$ in Eq. (83) was the value found by solving the differential equation (24). There is a second resonant layer, the value $\bar{y}_{R,2}$ of which is given by

$$\bar{y}_{R,2}=1-\bar{y}_{R,1} . \quad (100)$$

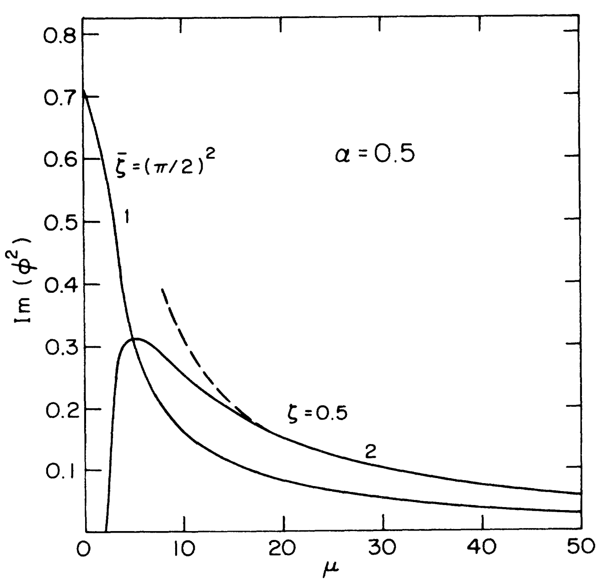

FIG. 3. Normalized growth rate $\mathrm{Im}(\phi^2)$ as a function of the gradient parameter μ for the fundamental symmetric solution: (curve 1) $\bar{\zeta}=(\pi/2)^2$; (curve 2) $\zeta=0.5$. The density profile is of a uniform gradient. The curves were obtained by numerical solution of Eq. (24). The dashed curve denotes values of the analytical expression (59). Here $\alpha=0.5$.

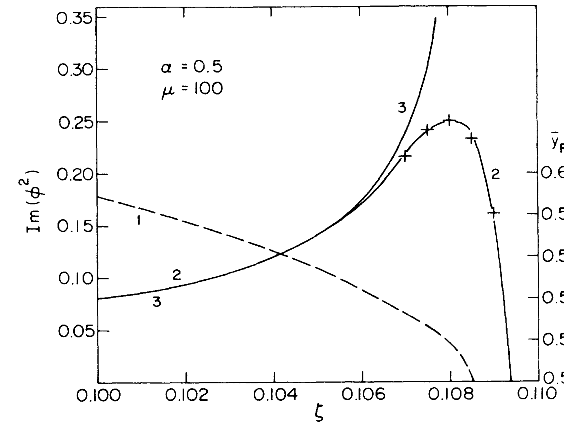

FIG. 4. Normalized growth rate $\mathrm{Im}(\phi^2)$ (solid lines) and the coordinate $\bar{y}_{R,1}$ of the resonant layer (dashed lines) as a function of the detuning parameter ζ for the fundamental antisymmetric solution. The density profile is parabolic [Eq. (99)]. (Curve 1) $\bar{y}_{R,1}$; (curve 2) $\mathrm{Im}(\phi^2)$ obtained by solving Eq. (24); (curve 3) $\mathrm{Im}(\phi^2)$ obtained from Eq. (82). The crosses denote values found by solving numerically Eq. (78). Here $\alpha=0.5$ and $\mu=100$. A second resonant layer is located at $\bar{y}_{R,2}=1-\bar{y}_{R,1}$.

It is clear that by increasing ζ one moves the resonant layers to the plane $\bar{y}=\frac{1}{2}$, and the growth rate increases. The reason becomes clear when one examines curve 3 which describes the value of $\mathrm{Im}(\phi^2)$ as found by the approximate formula (82). Near the plane $\bar{y}=\frac{1}{2}$ the derivative of f becomes smaller, the resonant layer is thicker, and the growth rate increases. Of course when the derivative $|f'(y_R)|$ is too small, Eq. (82) is not valid any more, and curves 2 and 3 do not coincide. The crosses show the values of the growth rate calculated by solving Eq. (78), and they agree with the solution of the differential equation (24). Thus in Fig. 4 we show the growth rate $\mathrm{Im}(\phi^2)$ calculated by three methods. Curve 2 results from the solution of the differential equation (24) and the crosses from solving Eq. (78). Since α is small the two methods yield similar results. Curve 3 shows the growth rate as calculated by the approximate formula (82) and the values found agree with those found by the two other methods as long as $|f'(y_R)|$ is not too small. Note also that when ζ is so large that there is no resonant layer, no value of $\bar{y}_R$ satisfies Eq. (83), the growth rate decreases abruptly to zero.

Cases in which the density inhomogeneity causes large gain degradation are not frequent and condition (95) is usually satisfied. We present here an example in which the gain is reduced as a result of density inhomoganeity. A sheet electron beam of thickness 0.2 cm ($a=0.1$ cm) and width 0.6 cm propagates along a wiggler of 500 G and wave number $k_0=6$ cm^{-1}. The electron beam current is 240 A and the current density 2 kA/cm^2. The beam energy is 360 kV. The resonant wavelength is about 0.2 cm. The parameter α is 9.7×10^{-4}, and the gain for a uniform beam [see Eqs. (25a) and (37)] is 0.031 cm^{-1}. If $\hat{\mu}/h_0$ is 1, the parameter μ is 0.252 and the gain is reduced to [Eq. (69)] 0.024 cm^{-1}. If the wiggler intensity is 250 G, the parameter α is 2.4×10^{-4}. The gain is then reduced from 0.016 to 0.006 cm^{-1}. On the other hand, for wiggler intensity of 1 kG the reduction is negligible.

The Raman regime is characterized by relatively high electron densities and low wiggler fields. The gain in the Raman regime is not sensitive to density nonuniformities if the parameters are such that the FEL is only mildly in the Raman regime, as required by (95). The wiggler intensity has to be not too low and the electron density not too high. The above numerical example illustrates this fact. The gain reduction in that example was bigger for lower wiggler fields. Even among FEL's which operate in the Raman regime the wiggler field is usually not too low, and condition (95) is usually satisfied. The conclusion of this study is, therefore, that density nonuniformities do not usually severely degrade the gain.

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